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LETTER TO THE EDITOR

Notes on the structure of the δ-function interacting gas. Intertwining operator in the degenerate affine Hecke algebra

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Abstract. The wavefunction of the δ-function interacting Bose gas on the infinite interval is studied. By introducing the intertwining operator of the degenerate affine Hecke algebra, the non-symmetric eigenfunction of the Dunkl operator is constructed.

1. Introduction

The one-dimensional quantum N-body Bose gas with a δ-function potential is an old problem, and has received much attention since the 1960s [1–3]. The Hamiltonian is given by

$$\mathcal{H} = -\sum_{i=1}^{N} \partial_i^2 + 2c \sum_{1\leqslant i<j\leqslant N} \delta(x_i - x_j) \tag{1}$$

where $\partial_i = \frac{\partial}{\partial x_i}$, and c is a constant. We note that there are no bound states when the interaction is repulsive $c > 0$. The nonlinear Schrödinger (NLS) model is integrable and exactly solvable. In solving the NLS model, the boundary condition becomes important. When the system is periodic [1–3], we can apply both the Bethe ansatz method and the finite-size corrections with help of the conformal field theory, and consequently obtain the critical exponents of the correlation functions [4, 5]. On the other hand, the situation differs when the system has an infinite volume [6]. It has also appeared [7, 8] that in an infinite volume a certain set of the differential-difference operators helps us to investigate the algebraic structure of the model (1). This set of operators was originally introduced in studies of the one-dimensional quantum integrable systems with inverse square interactions (the Calogero–Sutherland–Moser (CSM) model) [9–11], and has recently [12] been called the Dunkl operator. The simultaneous eigenfunctions of the CSM's Dunkl operators are called the *non-symmetric* Jack polynomials [13–15], and it is known that the eigenfunction of the CSM model is given by the *symmetric* Jack polynomials, which are the symmetrization of the *non-symmetric* Jack polynomials.

In this letter, motivating the result in the case of the CSM model, we shall construct the *non-symmetric* eigenfunction of the δ-function interacting Bose gas (1). We give two representations for the degenerate affine Hecke algebra, and introduce the 'intertwining

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operator'. As the Dunkl operator for the NLS model is intertwined with a partial differential operator (a momentum operator), one sees that the eigenfunction is not polynomial but a superposition of plain waves.

2. Degenerate affine Hecke algebra

We introduce the Dunkl operator $\hat{d}_i$ ($i = 1, 2, \ldots, N$) for the NLS model (1) [7, 8] as

$$\hat{d}_i = -\mathrm{i}\partial_i + \mathrm{i}\frac{c}{2}\sum_{j<i}(\varepsilon(x_i - x_j) - 1)\hat{s}_{i,j} + \mathrm{i}\frac{c}{2}\sum_{j>i}(\varepsilon(x_i - x_j) + 1)\hat{s}_{i,j}. \tag{2}$$

Here a function $\varepsilon(x)$ denotes a signature of x,

$$\varepsilon(x) = \begin{cases} +1 & \text{for } x > 0 \\ -1 & \text{for } x < 0. \end{cases} \tag{3}$$

Operator $\hat{s}_{i,j}$ exchanges coordinates of the ith and the jth particles, and satisfies

$$x_i \hat{s}_{i,j} = \hat{s}_{i,j} x_j.$$

For our latter convention we set $\hat{s}_j \equiv \hat{s}_{j,j+1}$ for $j = 1, 2, \ldots, N-1$. One sees that these operators satisfy the following identities;

$$[\hat{d}_i, \hat{d}_j] = 0 \tag{4a}$$

$$\hat{s}_j^2 = \mathbb{1} \tag{4b}$$

$$\hat{s}_j \hat{s}_{j+1} \hat{s}_j = \hat{s}_{j+1} \hat{s}_j \hat{s}_{j+1} \tag{4c}$$

$$[\hat{d}_i, \hat{s}_j] = 0 \qquad \text{for } i \neq j, j+1 \tag{4d}$$

$$\hat{s}_j \hat{d}_j - \hat{d}_{j+1} \hat{s}_j = \mathrm{i}c. \tag{4e}$$

These relations indicate that the operators $\{\hat{d}_i, \hat{s}_j | 1 \leqslant i \leqslant N; 1 \leqslant j \leqslant N-1\}$ represent the degenerate affine Hecke algebra defined by Drinfeld [16].

From the commutativity of the Dunkl operators (4*a*), we can define the quantum integrals of motion by

$$\mathcal{I}_n = \sum_{i=1}^{N} \pi(\hat{d}_i^n) \tag{5}$$

where $\pi(\cdot)$ indicates a projection onto a symmetric space, i.e. a bosonic space. The lowest three conserved operators are computed as follows [8]

$$\mathcal{I}_1 = \sum_{i=1}^{N} (-\mathrm{i}\partial_i) \tag{6a}$$

$$\mathcal{I}_2 = \mathcal{H} \tag{6b}$$

$$\mathcal{I}_3 = \sum_{i=1}^{N} (-\mathrm{i}\partial_i)^3 + 3c \sum_{1 \leqslant i < j \leqslant N} \delta(x_i - x_j)(-\mathrm{i}\partial_i - \mathrm{i}\partial_j). \tag{6c}$$

See that the Hamiltonian of the NLS model (1) is given from the Dunkl operators, and that the integrals of motion $\mathcal{I}_n$ commute with $\mathcal{H}$. This fact proves the quantum integrability of the NLS model (1) in the Liouville sense.

As a preparation in constructing simultaneous eigenfunctions of the Dunkl operators $\hat{d}_i$ (2), we introduce the integral operators $\hat{Q}_i$ [17] ($1 \leqslant i \leqslant N-1$) acting on arbitrary functions $f(x_1, \ldots, x_N)$ as

$$(\hat{Q}_i f)(\ldots, x_i, x_{i+1}, \ldots) = f(\ldots, x_{i+1}, x_i, \ldots) - c \int_0^{x_i - x_{i+1}} f(\ldots, x_i - t, x_{i+1} + t, \ldots)\, \mathrm{d}t. \tag{7}$$

The partial differential operators $-\mathrm{i}\partial_i$ and the integral operators $\hat{Q}_j$ satisfy the following relations;

$$[-\mathrm{i}\partial_i, -\mathrm{i}\partial_j] = 0 \tag{8a}$$

$$\hat{Q}_j^2 = \mathbb{1} \tag{8b}$$

$$\hat{Q}_j \hat{Q}_{j+1} \hat{Q}_j = \hat{Q}_{j+1} \hat{Q}_j \hat{Q}_{j+1} \tag{8c}$$

$$[-\mathrm{i}\partial_i, \hat{Q}_j] = 0 \qquad \text{for } i \neq j, j+1 \tag{8d}$$

$$\hat{Q}_j(-\mathrm{i}\partial_j) - (-\mathrm{i}\partial_{j+1})\hat{Q}_j = \mathrm{i}c. \tag{8e}$$

One finds that the operators $\{-\mathrm{i}\partial_i, \hat{Q}_j | 1 \leqslant i \leqslant N; 1 \leqslant j \leqslant N-1\}$ also constitute the degenerate affine Hecke algebra.

As a result, we have two representations for the degenerate affine Hecke algebra, (4) and (8), and there is a correspondence as follows

$$\left.\begin{matrix} \hat{d}_i \\ \hat{s}_j \end{matrix}\right\} \Longleftrightarrow \left\{\begin{matrix} -\mathrm{i}\partial_i \\ \hat{Q}_j. \end{matrix}\right. \tag{9}$$

In the next section, we shall diagonalize the Dunkl operators $\hat{d}_i$ (2). To this end, we shall introduce the intertwining operator which maps the *non-local* differential-difference operator $\hat{d}_i$ onto the *local* differential operator $-\mathrm{i}\partial_i$.

3. Eigenfunction

We shall diagonalize the Dunkl operators $\hat{d}_i$ (2) with a non-symmetric function $\psi(x)$;

$$\hat{d}_i \psi(x) = k_i \psi(x) \qquad \text{for } i = 1, 2, \ldots, N \tag{10}$$

where k_i corresponds to the quasimomentum of the ith particle. We assume that the wavefunction $\psi(x) \equiv \psi(x_1, \ldots, x_N)$ is continuous in $x \in \mathbb{R}^N$. We note that the eigenfunction $\psi(x)$ is in fact *non-symmetric* in its arguments x, and that the eigenfunction $\Psi(x)$ of the NLS model (1) is then given by symmetrizing $\psi(x)$;

$$\Psi(x) = \mathrm{Sym}(\psi(x)). \tag{11}$$

As a function $\psi(x)$ satisfies the eigenvalue problem (10) with the Dunkl operator (2), the *symmetric* eigenfunction $\Psi(x)$ becomes a simultaneous eigenfunction of the quantum integrals of motion $\mathcal{I}_n$ (5),

$$\mathcal{I}_n \Psi(x) = E_n \Psi(x) \tag{12a}$$

$$E_n = \sum_{i=1}^{N} k_i^n. \tag{12b}$$

We first consider the two-body case $N = 2$ for simplicity. We set the *non-symmetric* eigenfunction $\psi(x_1, x_2)$ of $\hat{d}_1$ and $\hat{d}_2$ as

$$\psi(x_1, x_2) = \theta(x_1 < x_2)\psi_1(x_1, x_2) + \theta(x_2 < x_1)\psi_2(x_1, x_2) \tag{13}$$

where $\theta(X)$ denotes

$$\theta(X) = \begin{cases} 1 & \text{if } X \text{ is true} \\ 0 & \text{if } X \text{ is false.} \end{cases}$$

By substituting (13) into eigenvalue problems (10), we obtain [8] that each function ψ_1 and ψ_2 has a form,

$$\psi_1(x_1, x_2) = \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2} \tag{14a}$$

$$\psi_2(x_1, x_2) = \frac{k_1 - k_2 - \mathrm{i}c}{k_1 - k_2}\mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2} + \frac{\mathrm{i}c}{k_1 - k_2}\mathrm{e}^{\mathrm{i}k_1x_2+\mathrm{i}k_2x_1}. \tag{14b}$$

The purpose of this letter is based on the observation that the two-body wavefunction $\psi(x_1, x_2)$ is written in a simple form with the integral operator $\hat{Q}_i$ (7) as

$$\psi(x_1, x_2) = (\theta(x_1 < x_2) + \theta(x_2 < x_1)\hat{s}_1\hat{Q}_1)\mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2}. \tag{15}$$

As a generalization to the N-body case, we find that the eigenfunction of the Dunkl operators is given by

$$\psi(x) = \hat{V}\exp\left(\sum_{i=1}^{N}\mathrm{i}k_ix_i\right) \tag{16}$$

where $\hat{V}$ is called the intertwining operator defined by

$$\hat{V} = \sum_{w\in S_N}\theta(x_{w^{-1}(1)} < \cdots < x_{w^{-1}(N)})\hat{s}_{w^{-1}}\hat{Q}_w. \tag{17}$$

Here w is the reduced decomposition in terms of the elementary transposition of each element of S_N, and $\hat{s}_{w^{-1}}$ and $\hat{Q}_w$ respectively denotes as

$$\hat{s}_{w^{-1}} = \hat{s}_{i_p}\ldots\hat{s}_{i_2}\hat{s}_{i_1} \qquad \hat{Q}_w = \hat{Q}_{i_1}\hat{Q}_{i_2}\ldots\hat{Q}_{i_p}$$

where $1 \leqslant i_1, i_2, \ldots, i_p \leqslant N - 1$. This form of the wavefunction shows that the integral operator $\hat{Q}_i$ (7) represents the scattering matrix between the ith and $(i+1)$th particles, and the braid relation (8*c*) is based on that the scattering matrix of the NLS model satisfies the Yang–Baxter relation which indicates the integrability of the model [18].

The fact that a function (16) becomes an eigenfunction of the Dunkl operator could be given by proving an identity,

$$\hat{d}_i\hat{V} = \hat{V}(-\mathrm{i}\partial_i). \tag{18}$$

The operator $\hat{V}$ intertwines the two representations of the degenerate affine Hecke algebra, (4) and (8). We thus obtained the operator $\hat{V}$, which intertwines the δ-function gas and free particle systems. The proof of (18) is rather straightforward. We recall that the Dunkl operator $\hat{d}_i$ (2) is written as

$$\hat{d}_\xi = -\mathrm{i}\partial_\xi - \mathrm{i}c\sum_{\alpha>0}(\xi, \alpha)\hat{s}_\alpha\theta((x, \alpha) < 0). \tag{2$'$}$$

Here we use $\xi = \epsilon_i$ for $1 \leqslant i \leqslant N$, and $x = \sum_{i=1}^{N} x_i\epsilon_i$ with bases ϵ_i of $\mathbb{R}^N$. A set of positive roots R_+ is defined as $R_+ = \{\epsilon_i - \epsilon_j | 1 \leqslant i < j \leqslant N\}$, and $\alpha > 0$ means $\alpha \in R_+$. The inner product is defined as $(\epsilon_i, \epsilon_j) = \delta_{ij}$. As the operators $\hat{Q}_i$ and $-\mathrm{i}\partial_j$ constitute the degenerate affine Hecke algebra, we have [13]

$$\hat{Q}_w(-\mathrm{i}\partial_\xi)\hat{Q}_{w^{-1}} = -\mathrm{i}\partial_{w\xi} - \mathrm{i}c\sum_{\substack{\alpha>0 \\ w^{-1}\alpha<0}}(w\xi, \alpha)\hat{Q}_\alpha. \tag{19}$$

Further, we note that

$$(\partial_i\theta)(x_1 < x_2 < \cdots < x_N) = (\delta(x_i - x_{i-1}) - \delta(x_i - x_{i+1}))\theta(\ldots < x_{i-1} < x_{i+1} < \cdots) \tag{20}$$

$$\delta(x_i - x_{i+1})\hat{Q}_i = \hat{s}_i. \tag{21}$$

Using relations (19)–(21) and a definition (2′) of the Dunkl operator, we obtain an intertwining relation (18).

As we find that the Dunkl operator $\hat{d}_i$ (2) is intertwined with a momentum operator $-\mathrm{i}\partial_i$ by the operator $\hat{V}$ (18), the *non-symmetric* wavefunction $\psi(x)$ is given by a superposition of plain waves, $\exp(\sum_i \mathrm{i}k_i x_i)$. We note that the symmetrized eigenfunction $\Psi(x)$ (11), as an eigenfunction of the conserved operators $\mathcal{I}_n$ (12), is then given by

$$\begin{aligned}\Psi(x) &= \sum_w \hat{Q}_w \exp\left(\sum_{i=1}^N \mathrm{i}k_i x_i\right) \\ &= \sum_w c(wk) \exp\left(\sum_{i=1}^N \mathrm{i}k_{w_i} x_i\right)\end{aligned} \tag{22}$$

where

$$c(k) = \prod_{\alpha\in R_+} \frac{(k,\alpha) + \mathrm{i}c}{(k,\alpha)}.$$

See the appendix for explicit forms of the wavefunctions $\psi(x)$ and $\Psi(x)$ up to $N = 3$.

4. Concluding remarks

We have defined the intertwining operator $\hat{V}$ (18) for the δ-function interacting gas; the operator $\hat{V}$ intertwines the Dunkl operator $\hat{d}_j$ and the partial differential operator $-\mathrm{i}\partial_j$, which are two different representations of the degenerate affine Hecke algebra as shown in (4) and (8).

We recall that the intertwining operator for the CSM model was studied in [19];

$$\hat{T}_i \hat{V}_{\mathrm{C}} = \hat{V}_{\mathrm{C}} \partial_i \tag{23}$$

where $\hat{T}_i$ is the Dunkl operator for the rational Calogero model,

$$\hat{T}_i = \partial_i + c \sum_{\substack{j=1 \\ j\neq i}}^N \frac{1}{x_i - x_j}(1 - \hat{s}_{i,j}). \tag{24}$$

The explicit form of the intertwining operator $\hat{V}_{\mathrm{C}}$ is rather complicated [19]. It should be noted that, while our Dunkl operators $\hat{d}_i$ (2) for the NLS model constitute the degenerate affine Hecke algebra, the original Dunkl operator $\hat{T}_i$ is not a representation of the degenerate affine Hecke algebra.

The eigenfunctions for the NLS model associated with the root systems of type-B and type-G [20, 8, 21] could be constructed in the same manner. The lattice analogue of the NLS model [22] could be studied by considering a deformation of the integral operator (7).

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Appendix. Explicit form of eigenfunctions

To clarify the notations in this letter, we give the explicit wavefunctions $\psi(x)$ and $\Psi(x)$ for simple cases, $N = 2$ and $N = 3$. The *non-symmetric* wavefunction $\psi(x)$ is an eigenfunction of the Dunkl operator $\hat{d}_i$ (2) as in (10), and the *symmetric* function $\Psi(x)$ is a symmetrization of $\psi(x)$ and becomes an eigenfunction of the bosonic NLS model (1). Hereafter we use a scattering function $S(i, j)$ and the plain wave $\chi(i_1, i_2, i_3, \ldots)$ as

$$S(i, j) = \frac{k_i - k_j + \mathrm{i}c}{k_i - k_j} \tag{A1}$$

$$\chi(i_1, i_2, i_3, \ldots) = \exp(\mathrm{i}k_1 x_{i_1} + \mathrm{i}k_2 x_{i_2} + \mathrm{i}k_3 x_{i_3} + \cdots). \tag{A2}$$

(i) $N = 2$

- non-symmetric function;

$$\psi(x) = (\theta(x_1 < x_2) + \theta(x_2 < x_1)\hat{s}_1 \hat{Q}_1)\chi(1, 2).$$

- Symmetric function;

$$\Psi(x) = S(1, 2)\chi(1, 2) + S(2, 1)\chi(2, 1).$$

(ii) $N = 3$

- non-symmetric function;

$$\begin{aligned}\psi(x) = (\theta(x_1 < x_2 < x_3) &+ \theta(x_2 < x_1 < x_3)\hat{s}_1 \hat{Q}_1 + \theta(x_1 < x_3 < x_2)\hat{s}_2 \hat{Q}_2 \\ &+ \theta(x_2 < x_3 < x_1)\hat{s}_1 \hat{s}_2 \hat{Q}_2 \hat{Q}_1 + \theta(x_3 < x_1 < x_2)\hat{s}_2 \hat{s}_1 \hat{Q}_1 \hat{Q}_2 \\ &+ \theta(x_3 < x_2 < x_1)\hat{s}_1 \hat{s}_2 \hat{s}_1 \hat{Q}_1 \hat{Q}_2 \hat{Q}_1)\chi(1, 2, 3).\end{aligned}$$

- Symmetric function;

$$\begin{aligned}\Psi(x) = S(1, 2)S(1, 3)S(2, 3)\chi(1, 2, 3) &+ S(2, 1)S(1, 3)S(2, 3)\chi(2, 1, 3) \\ &+ S(1, 2)S(1, 3)S(3, 2)\chi(1, 3, 2) + S(2, 1)S(3, 1)S(2, 3)\chi(3, 1, 2) \\ &+ S(1, 2)S(3, 1)S(3, 2)\chi(2, 3, 1) + S(2, 1)S(3, 1)S(3, 2)\chi(3, 2, 1).\end{aligned}$$

References

[1] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312
[2] Lieb E H and Liniger W 1963 *Phys. Rev.* **130** 1605
[3] Flicker M and Lieb E H 1967 *Phys. Rev.* **161** 179
[4] Kawakami N and Yang S-K 1991 *Prog. Theor. Phys. Suppl.* **106** 157
[5] Schlottmann P 1994 *J. Phys.: Condens. Matter* **6** 1359
[6] Faddeev L D 1980 *Sov. Sci. Rev. Sec. C: Math. Phys.* **1** 107
[7] Murakami S and Wadati M 1996 *J. Phys. A: Math. Gen.* **29** 7903
[8] Komori Y and Hikami K 1997 *Int. J. Mod. Phys.* A **12** 5397
[9] Polychronakos A P 1992 *Phys. Rev. Lett.* **69** 703
[10] Bernard D, Gaudin M, Haldane F D M and Pasquier V 1993 *J. Phys. A: Math. Gen.* **26** 5219
[11] Hikami K 1996 *J. Phys. A: Math. Gen.* **29** 2135
[12] Dunkl C F 1989 *Trans. Am. Math. Soc.* **311** 167
[13] Opdam E M 1995 *Acta Math.* **175** 75
[14] Cherednik I 1994 *Math. Res. Lett.* **1** 319
[15] Macdonald I G 1994–95 *Séminaire Bourbaki* **797** 1
[16] Drinfeld V G 1986 *Funct. Annal. Appl.* **20** 58
[17] Heckman G J and Opdam E M 1997 *Ann. Math.* **145** 139
[18] Zamolodchikov A B and Zamolodchikov A B 1979 *Ann. Phys.* **120** 253
[19] Dunkl C F 1995 *Trans. Am. Math. Soc.* **347** 3347
Dunkl C F *Monatsh. Math.* to appear

[20] Gaudin M 1971 *Phys. Rev.* A **4** 386
[21] Komori Y and Hikami K 1997 *J. Phys. A: Math. Gen.* **30** 1913
[22] Hikami K and Murakami S 1996 *Phys. Lett.* A **221** 109